Mon. Not. R. Astron. Soc. **000**, 000–000 (0000) Printed 1 February 2008

AN ANALYTIC APPROXIMATION TO THE ISOTHERMAL SPHERE

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1 February 2008

ABSTRACT

We present a useful analytic approximation to the solution of the Lane-Emden equation for infinite polytropic index - the isothermal sphere. The optimized expression obtained for the density profile can be accurate to within 0.04% within 5 core-radii and to 0.1% within 10 core-radii.

Key words: galaxy models – galaxy structure – polytropic solutions

1 INTRODUCTION

We construct an analytic approximation to the full non-singular isothermal sphere. The approximations currently in use provide a good fit either to the inner regions ($r \leq 2\,r_{\rm core}$) or to the asymptotic behavior. The Lane-Emden equation for the gaseous polytrope with polytropic index $n \to \infty$ is identical to that of a self-gravitating isothermal sphere. All polytropes with $n \geq 5$ are infinite and hence no analytic solutions exist. In a recent paper, Liu (1996) has exhaustively examined approximate analytic solutions for polytropes with general index, where he obtains a solution for the isothermal case to within $< 1\%$. In this brief note, we report a simpler analytic form, accurate to within 0.04% within 5 core-radii.

2 THE ISOTHERMAL SPHERE

The Lane-Emden equation written in terms of the standard variables (see Emden 1907) is,

$$\frac{d^2w}{d\xi^2} + \frac{2}{\xi}\frac{dw}{d\xi} = e^{-w}, \qquad (1)$$

where $w = \ln \rho_0/\rho$; $\xi = r/r_0$; $r_0^2 = \sigma^2/4\pi G\rho_0$; σ is the constant velocity dispersion and r_0 the core radius.

The solution for the density distribution can be expanded into a series,

$$\frac{\rho}{\rho_0} = 1 - (\frac{1}{6})\xi^2 + (\frac{1}{45})\xi^4 - ... \tag{2}$$

The approximation that is used often in the literature (which we denote by the subscript (a) henceforth) follows from truncating equation (2) at the second order.

$$\rho(\xi)_a = \frac{1}{1+\frac{\xi^2}{6}}. \tag{3}$$

While the mathematical form is simple, this profile overestimates mass outside r_0 and differs from the exact solution by a factor of 3 as $r \to \infty$.

As an ansatz, we attempt the following analytic form for the approximation:

$$\rho(\xi)_{\rm approx} = \frac{A}{a^2+\xi^2} - \frac{B}{b^2+\xi^2}, \tag{4}$$

where A, B, a^2 and b^2 are to be determined. The solution obtained (see Appendix for details) is given by,

$$\rho(\xi)_b = \frac{5}{1+\frac{\xi^2}{10}} - \frac{4}{1+\frac{\xi^2}{12}}. \tag{5}$$

This expression is correct asymptotically and is accurate to within 1% up to $\xi = 5$. Preserving the general form, this approximation can be optimized further yielding,

$$a^2 = [10 - (\frac{5}{11}\delta - \frac{2}{11}\epsilon)] \ ; \ b^2 = [12 - (\frac{9}{11}\delta + \frac{3}{11}\epsilon)] \ ;$$

$$\frac{A}{a^2} = (5+\epsilon) \ ; \ \frac{B}{b^2} = (4+\epsilon), \tag{6}$$

where the additional parameters ϵ and δ have now been chosen (see Appendix for details) to optimize the degree of accuracy in terms of agreement with the full exact solution. This solution is plotted in Figures 1 and 2. The mathematical form of the optimized solution is convenient appropos Abel inversion since it can be inverted analytically. The corresponding projected quantities for our formula - the surface density and mass are easily computed and have the following analytic forms:

$$\Sigma(R) = \frac{A\pi}{\sqrt{a^2+R^2}} - \frac{B\pi}{\sqrt{b^2+R^2}}, \tag{7}$$

where R is the projected radius. These spherical models can be easily generalized to describe elliptical mass distributions as well, akin to those proposed by Kassiola, Kovner, & Fort

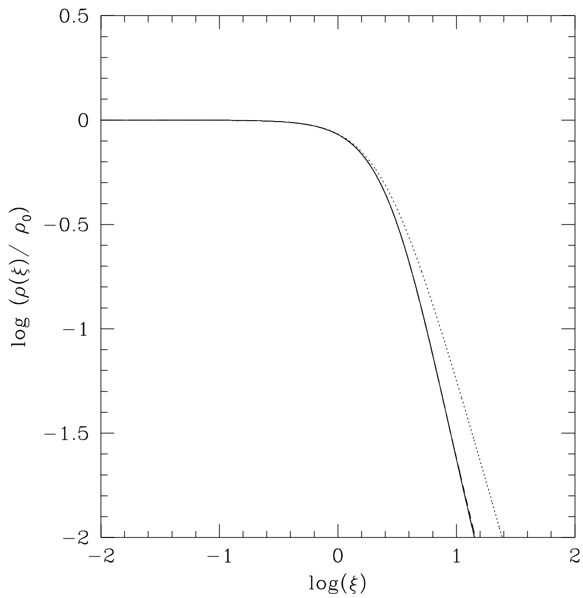

Figure 1. VARIOUS APPROXIMATIONS TO THE ISOTHERMAL SOLUTION: solid curve - exact solution, dotted curve - approx(a), dashed curve - approx(b), long-dashed curve - approx(c)

(1993).

$$M_{3\mathrm{D}}(\xi) \;=\; 4\pi\,[\,A\,a\,(\frac{\xi}{a}\;-\;\tan^{-1}(\frac{\xi}{a}))\;-\;B\,b\,(\frac{\xi}{b}\;-\{\tan^{-1}(\frac{\xi}{b}))], \tag{8}$$

The potential on the plane corresponding to the surface density Σ is,

$$\begin{aligned}\phi_{2D} \;=\; & A\pi[\sqrt{a^2+R^2} - a\ln R - a\ln(a^2 + a\sqrt{a^2+R^2})] \\ & - B\pi[\sqrt{b^2+R^2} - b\ln R - b\ln(b^2 + b\sqrt{b^2+R^2})].\end{aligned} \tag{9}$$

These projected quantities are of interest in many physical problems - for instance, in the context of modelling gravitational lensing observations. The two primary effects producing by lensing are the isotropic manginification and the anisotropic shear. The magnification κ produced by the potential is given by,

$$\kappa(R) \;=\; \kappa_0[\,\frac{A\pi}{\sqrt{a^2+R^2}}\;-\;\frac{B\pi}{\sqrt{b^2+R^2}}\,] \tag{10}$$

where $\kappa_0 \;=\; \frac{1}{\Sigma_{\rm crit}}$; and $\Sigma_{\rm crit}$ is the critical surface density given the geometrical configuration of the source, lens and the observer. And the induced shear γ is,

$$\begin{aligned}\gamma(R) \;\;=\;\; & \kappa_0[\,-\frac{A\pi}{\sqrt{R^2+a^2}} + \frac{2A\pi}{R^2}(\sqrt{R^2+a^2} - a) \\ + \;\; & \frac{B\pi}{\sqrt{R^2+b^2}} - \frac{2B\pi}{R^2}(\sqrt{R^2+b^2} - b)\,].\end{aligned} \tag{11}$$

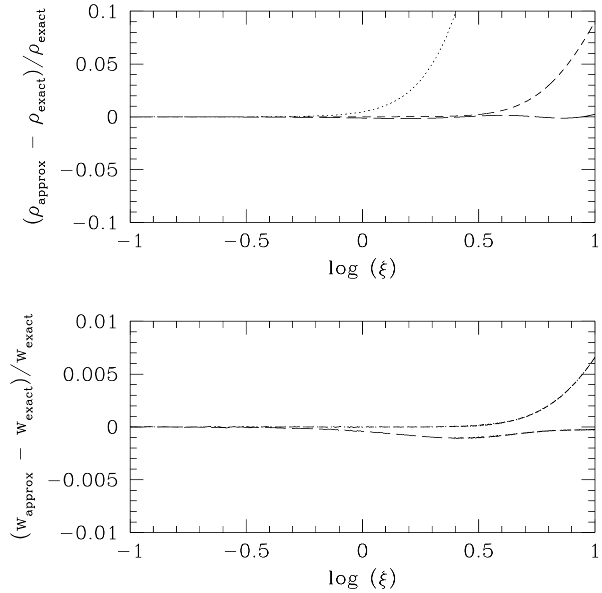

Figure 2. DIFFERENTIAL ERRORS: (i) TOP PANEL - density (dotted curve - approx(a), dashed curve - approx(b), long-dashed curve - approx(c)) and (ii) LOWER PANEL - the potential (dotted curve - approx (a), dashed curve - approx(b), long-dashed curve - approx(c)) (the dotted curve and dashed curve are coincident)

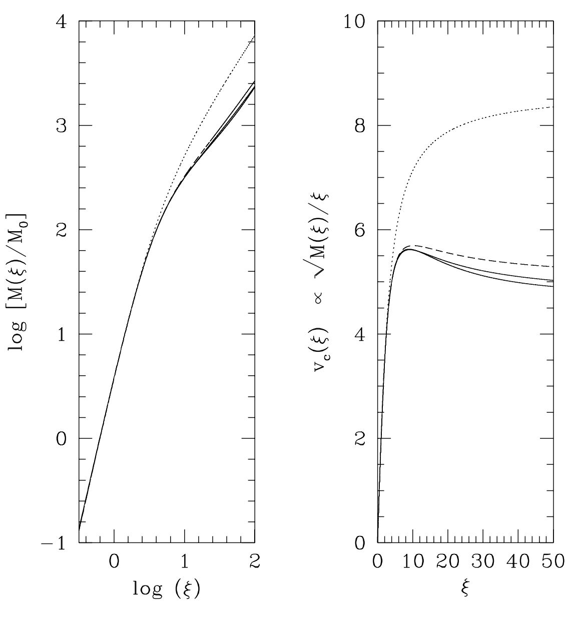

Figure 3. (i) LEFT PANEL: MASS ENCLOSED WITHIN r (solid curve - exact solution, dotted curve - approx(a), dashed curve - approx(b), long-dashed curve - approx(c)) and (ii) RIGHT PANEL: circular velocity profiles (solid curve - exact solution, dotted curve - approx(a), dashed curve - approx(b), long-dashed curve - approx(c))

3 CONCLUSIONS

The analytic approximation presented above is potentially useful in the context of many physical problems and is particularly useful since the projected quantities have simple ana-

lytic forms. We also point out that within 5 core-radii, while the analytic approximation to the isothermal sphere currently in use systematically over-estimates the mass enclosed our formula is accurate to within 0.04%.

4 APPENDIX

4.1 Fitting to the exact solution

Starting with our ansatz for the functional form,

$$\rho(\xi)_{\mathrm{approx}} = \frac{A}{a^2+\xi^2} - \frac{B}{b^2+\xi^2}. \tag{12}$$

Expanding the above as,

$$\rho(\xi)_{\mathrm{approx}} = \frac{A}{a^2}\left(\frac{1}{1+\frac{\xi^2}{a^2}}\right) - \frac{B}{b^2}\left(\frac{1}{1+\frac{\xi^2}{b^2}}\right), \tag{13}$$

$$\begin{aligned}\rho(\xi)_{\mathrm{approx}} &= \frac{A}{a^2}\left(1 - \frac{\xi^2}{a^2} + \frac{\xi^4}{a^4} + ...\right) \\ &- \frac{B}{b^2}\left(1 - \frac{\xi^2}{b^2} + \frac{\xi^4}{b^4} + ...\right),\end{aligned} \tag{14}$$

Comparing terms to corresponding orders in ξ in equations 1 and 2, we obtain the following system of equations:

$$\frac{A}{a^2} - \frac{B}{b^2} = 1\ ;\ \frac{A}{a^4} - \frac{B}{b^4} = \frac{1}{6}\ ;\ \frac{A}{a^6} - \frac{B}{b^6} = \frac{1}{45}. \tag{15}$$

Requiring the asymptotic behavior as $\xi \rightarrow \infty$ to match up to the full solution, we obtain the additional equation to close the above system of 4 simultaneous equations in 4 unknowns.

$$A - B = 2, \tag{16}$$

we obtain an exact solution!,

$$\rho(\xi)_b = \frac{5}{1+\frac{\xi^2}{10}} - \frac{4}{1+\frac{\xi^2}{12}}. \tag{17}$$

4.2 Optimized approximation

We introduce additional parameters ϵ, δ, x and y and constrain their values to the desired degree of accuracy as follows,

$$\rho(\xi)_c = \frac{5+\epsilon}{1+\frac{\xi^2}{10-x\epsilon}} - \frac{4+\epsilon}{1+\frac{\xi^2}{12-y\epsilon}}. \tag{18}$$

We require agreement asymptotically as $\xi \rightarrow \infty$, therefore simplifying the equation above for large ϵ,

$$(5+\epsilon)(10-x\epsilon) \;-\; (4+\epsilon)(12-y\epsilon) \;=\; 2, \tag{19}$$

ignoring terms of $O(\epsilon^2)$,

$$x \;=\; (\frac{5}{9}y \;-\; \frac{1}{3}). \tag{20}$$

Substituting the above back into equation (16),

$$\rho(\xi)_c \;=\; \frac{5\;+\;\epsilon}{1+\frac{\xi^2}{10-x\epsilon}} \;-\; \frac{4\;+\;\epsilon}{1+\frac{\xi^2}{12-(\frac{1}{2}+\frac{5}{4}x\epsilon)}}. \tag{21}$$

Now expanding the above two terms on the RHS into a series and matching the corresponding terms of the same order in ξ to $O(\epsilon)$ and matching asymptotically to δ degree of accuracy we have,

$$y \;=\; (\frac{3}{11} \;+\; \frac{9}{11}\frac{\delta}{\epsilon}). \tag{22}$$

For an RMS error of 0.04% to $\xi \;=\; 5$, we find $\epsilon \;=\; -0.07$ and $\delta \;=\; -0.269$. If we do not treat ϵ as small and do not demand the correct asymptotic solution at large ξ, we can obtain a solution with RMS error of 0.1% at $\xi \;=\; 10$. This is $\epsilon \;=\; -0.635$ and $\delta \;=\; -0.6$ which is used in the plots. In this note, we have approximated the full untruncated isothermal sphere and have not addressed the problems of truncating it.

ACKNOWLEDGMENTS

PN acknowledges funding from the Isaac Newton Studentship and Trinity College at the University of Cambridge.

REFERENCES

Emden R., 1907, in Gaskugeln - Anwendungen der Mechan. Warmtheorie. Druck und Verlag Von B. G. Teubner, Leipzig, p. 129

Kassiola A., Kovner I., Fort B., 1993, ApJ, 400, 41

Liu F. K., 1996, MNRAS, 281, 1197